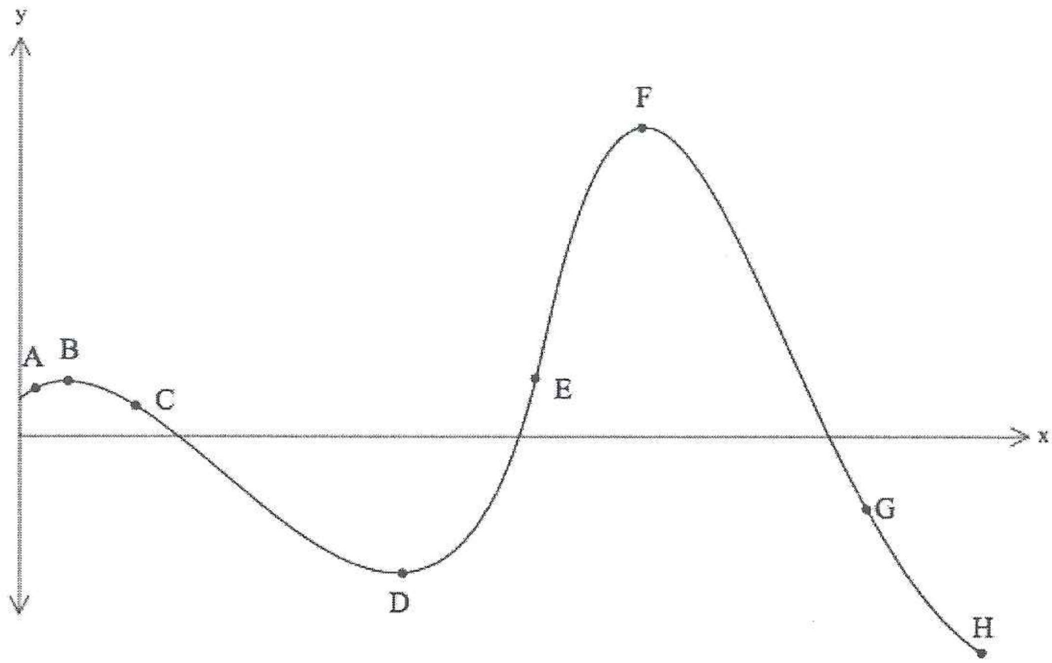
Solutions

Topic 7: Introduction to differential calculus

Topic number	Contents
7.1	Concept of the derivative as a rate of change. Tangent to a curve.
7.2	The principle that $$f(x) = ax^n \Rightarrow f'(x) = anx^{n-1}$$ The derivative of functions of the form $f(x) = ax^n + bx^{n-1} + \cdots$ where all exponents are integers.
7.3	Gradients of curves for given values of x. Values of x where $f'(x)$ is given. Equation of the tangent at a given point. Equation of the line perpendicular to the tangent at a given point(normal).
7.4	Increasing and decreasing functions. Graphical interpretation of $f'(x) > 0, f'(x) = 0$ and $f'(x) < 0$.
7.5	Values of x where the gradient of a curve is zero. Solution of $f'(x) = 0$ Stationary points. Local maximum and minimum points.
7.6	Optimization problems

1. Consider the graph of the function $y = f(x)$ defined below.



Write down **all** the labeled points on the curve

a) That are local maximum points; B, F (1 mark)

b) Where the function attains its least value; H (1 mark)

c) Where the function attains its greatest value; F (1 mark)

d) Where the gradient of the tangent to the curve is positive; (1 mark)

(increasing) A E

e) Where $f(x) > 0$ and $f'(x) < 0$. (2 marks)

graph is positive (above x-axis) decreasing C

2. Consider the curve $y = x^2$.

a) Write down $\frac{dy}{dx}$. (1 mark)

$\frac{dy}{dx} = 2x$

The point $P(3, 9)$ lies on the curve $y = x^2$.

b) Find the gradient of the tangent to the curve at P. (2 marks)

$f'(x) = 2x$

$f'(3) = 2 \cdot 3 = \boxed{6}$

c) Find the equation of the normal to the curve at P. Give your answer in the form $y = mx + b$ (3 marks)

if gradient is 6

slope of normal is $-\frac{1}{6}$

$(3, 9)$

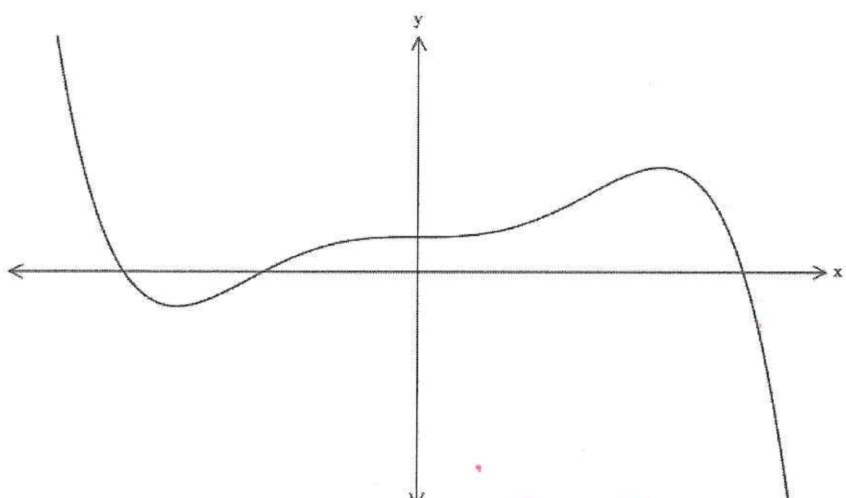
$y - 9 = -\frac{1}{6}(x - 3)$

$y - 9 = -\frac{1}{6}x + \frac{1}{2}$

$+9 \qquad +9$

$y = -\frac{1}{6}x + \frac{19}{2}$

3. A sketch of the function $f(x) = 5x^3 - 3x^5 + 1$ is shown for $-1.5 \leq x \leq 1.5$ and $-6 \leq y \leq 6$.



$f(x) = 5x^3 - 3x^5 + 1$

a) Write down $f'(x)$. (2 marks)

$f'(x) = 15x^2 - 15x^4$

b) Find the equation of the tangent to the graph of $y = f(x)$ at $(1, 3)$ (2 marks)

$f'(1) = 15(1)^2 - 15(1)^4 = 0$

$y - 3 = 0(x - 1)$

$y - 3 = 0$

$\boxed{y = 3}$

c) Write down the coordinates of the second point where this tangent intersects the graph of $y = f(x)$. (2 marks)

where does $5x^3 - 3x^5 + 1 = 3$

graph $y = 5x^3 - 3x^5 + 1$

$y = 3$

see where they intersect

$x = -1.38$

4. Let $f(x) = x^4$.

a) Write down $f'(x)$. (1 mark)

$f(x) = x^4$

$f'(x) = 4x^3$

Point $P(2, 16)$ lies on the graph of f.

b) Find the gradient of the tangent to the graph of $y = f(x)$ at P. (2 marks)

$f'(2) = 4(2)^3 = 32$

c) Find the equation of the normal to the graph at P. Give your answer in the form $ax + by + d = 0$, where a, b and d are integers. (3 marks)

if gradient is 32, normal gradient is $-\frac{1}{32}$

$y - 16 = -\frac{1}{32}(x - 2)$

to get rid of denominator, multiply both sides by -32

$-32(y-16) = \left[\cancel{-\frac{1}{32}}(x-2)\right]\cancel{-32}$

$-32y + 512 = x - 2$

$\boxed{0 = x + 32y - 514}$

5. Consider the curve $y = x^3 + kx$.

a) Write down $\frac{dy}{dx}$. (1 mark)

$y = x^3 + kx$

$\frac{dy}{dx} = 3x^2 + k$

The curve has a local minimum at the point where $x = 2$.

minimums when $\frac{dy}{dx} = 0$

b) Find the value of k. (3 marks)

$3x^2 + k = 0$ at $x = 2$

$3(2)^2 + k = 0$

$12 + k = 0$

$k = -12$

c) Find the value of y at this local minimum. (2 marks)

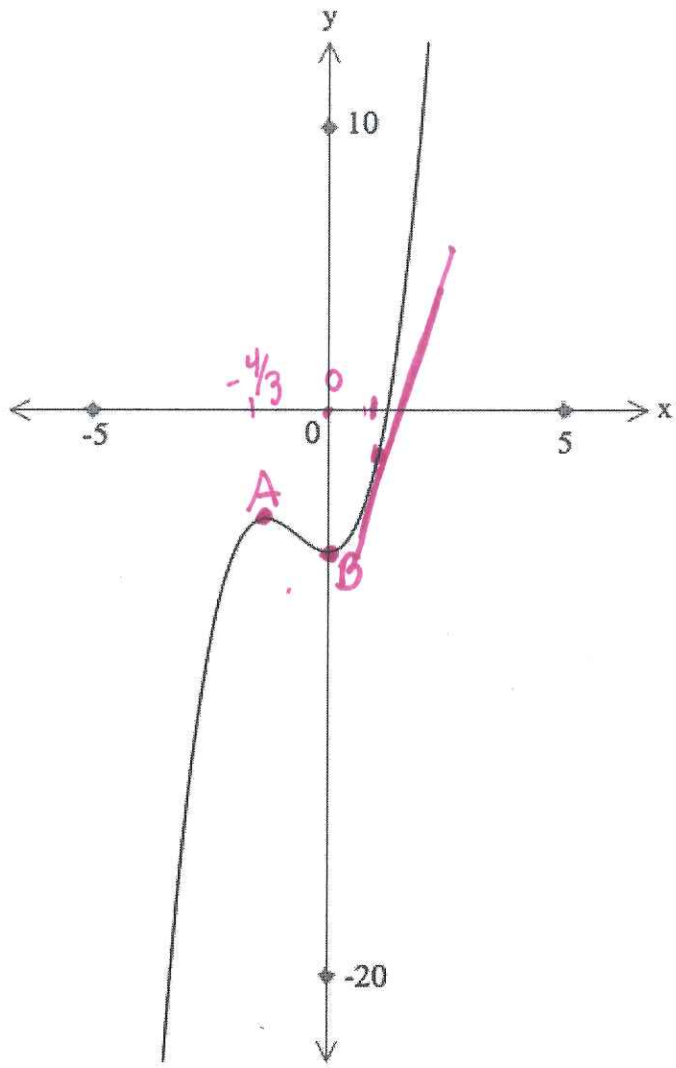
$y = x^3 + (-12)x$

$y = x^3 - 12x$

at $x = 2$ $y = 2^3 - 12(2)$

$= 8 - 24 = -16$

6. Consider the graph of the function $f(x) = x^3 + 2x^2 - 5$.



a) Label the local maximum as A on the graph. (1 mark)

b) Label the local minimum as B on the graph. (1 mark)

c) Write down the interval where $f'(x) < 0$. (1 mark)

$f'(x) = 3x^2 + 4x$ decreasing

$x(3x+4)$ $\quad x=0 \quad 3x+4=0 \quad x = -\frac{4}{3}$ $\quad \boxed{-\frac{4}{3} < x < 0}$

d) Draw the tangent to the curve at $x = 1$ on the graph, (1 mark)

$f(1) = 1^3 + 2(1)^2 - 5 = -2$

$(1, -2)$

e) Write down the equation of the tangent at $x = 1$. (2 marks)

$f'(1) = 3(1)^2 + 4(1) = 7$

$y - {-2} = 7(x-1)$

$y + 2 = 7x - 7$

$ -2 \qquad\quad -2$

$\boxed{y = 7x - 9}$

7. A function is given as $f(x) = 2x^3 - 5x + \frac{4}{x} + 3,\ -5 \le x \le 10,\ x \ne 0$

a) Write down the derivative of the function. (4 marks)

$f(x) = 2x3 - 5x + 4x^{-1} + 3$

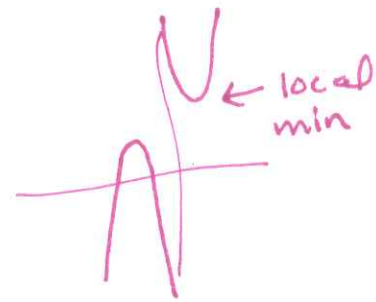
$f'(x) = 6x^2 - 5 - 4x^{-2}$ or $6x^2 - 5 - \frac{4}{x^2}$

b) Use your graphic display calculator to find the coordinates of the local minimum point of $f(x)$ in the given domain. (2 marks)

$y = 2x^3 - 5x + 4/x + 3$

2nd Calc (trace)

3: minimum



(move cursor →

to be close to min -

left of min enter

right of min enter

enter

$\boxed{x = 1.15 \quad y = 3.77}$